HDG Method for Time-dependent Problems

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Outline

Today lecture's main goals:

- Heat equation
- Wave equation
- Temporal discretization by BDF methods
- Temporal discretization by DIRK methods

Heat Equation

We consider the following heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla \cdot (\kappa \nabla u) &= f, && \text{in } \Omega \times (0,T], \\ u &= g_{\mathrm{D}}, && \text{on } \partial\Omega_{\mathrm{D}} \times (0,T], \\ \kappa \nabla u \cdot \boldsymbol{n} &= g_{\mathrm{N}}, && \text{on } \partial\Omega_{\mathrm{N}} \times (0,T], \\ u &= u_0, && \text{on } \Omega \times \{0\}. \end{aligned} \tag{1}$$

We rewrite it as

$$\begin{aligned} \boldsymbol{q} - \kappa \nabla u &= 0, && \text{in } \Omega \times (0,T], \\ \frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{q} &= f, && \text{in } \Omega \times (0,T], \\ u &= g_{\mathrm{D}}, && \text{on } \partial\Omega_{\mathrm{D}} \times (0,T], \\ \boldsymbol{q} \cdot \boldsymbol{n} &= g_{\mathrm{N}}, && \text{on } \partial\Omega_{\mathrm{N}}, \times (0,T], \\ u &= u_0, && \text{on } \Omega \times \{0\}. \end{aligned} \tag{2}$$

Here κ is diffusion coefficient.

Semi-Discrete Local Problem

Let $(\boldsymbol{q}_h(t), u_h(t)) \in [\mathcal{P}_k(K)]^d \times \mathcal{P}_k(K)$ be such that

$$\begin{aligned}(\kappa^{-1}\boldsymbol{q}_h, \boldsymbol{v})_K + (u_h, \nabla \cdot \boldsymbol{v})_K - \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0, \\ (\frac{\partial u_h}{\partial t}, w)_K + (\boldsymbol{q}_h, \nabla w)_K - \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w \rangle_{\partial K} &= (f, w)_K,\end{aligned} \tag{3}$$

for all $(\boldsymbol{v}, w) \in [\mathcal{P}_k(K)]^d \times \mathcal{P}_k(K)$ and for all $t \in (0, T]$, and that

$$(u_h(t=0), w)_K = (u_0, w)_K, \quad \forall w \in \mathcal{P}_k(K), \tag{4}$$

where

$$\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h - \tau(u_h - \widehat{u}_h)\boldsymbol{n}\ . \tag{5}$$

Here $\tau > 0$ is a stabilization parameter.

Semi-Discrete Global Problem

We find $\widehat{u}_h(t) \in M_h^k$ such that

$$\langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial\Omega} + \langle \widehat{u}_h - g_{\mathrm{D}}, \mu \rangle_{\partial\Omega_{\mathrm{D}}} + \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} - g_{\mathrm{N}}, \mu \rangle_{\partial\Omega_{\mathrm{N}}} = 0, \quad (6)$$

for all $\mu \in M_h^k$ and for all $t \in (0, T]$.

The global problem (6) enforces the boundary conditions and the jump condition in the flux.

Semi-Discrete Formulation

We find $(\boldsymbol{q}_h, u_h, \widehat{u}_h) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (u_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\
(\frac{\partial u_h}{\partial t}, w)_{\mathcal{T}_h} + (\boldsymbol{q}_h, \nabla w)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial\mathcal{T}_h} &= (f, w)_{\mathcal{T}_h},\\
\langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \langle \widehat{u}_h - g_{\mathrm{D}}, \mu\rangle_{\partial\Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}, \mu\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned} \tag{7}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ and for all $t \in (0, T]$, and that

$$
(u_h(t=0), w)_K = (u_0, w)_{\mathcal{T}_h}, \quad \forall w \in W_h^k, \tag{8}
$$

where

$$
\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h - \tau(u_h - \widehat{u}_h)\boldsymbol{n} \,. \tag{9}
$$

This completes the semi-discrete formulation.

Heat Equation: Temporal Discretization

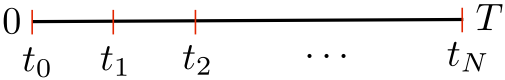
We divide the time domain $[0, T]$ into N intervals

$$[0, T] = [t_0, t_1] \times [t_1, t_2] \times \ldots \times [t_{N-1}, t_N]. \tag{10}$$

For simplicity, we assume that

$$t_n - t_{n-1} = \Delta t, \quad n = 1, \ldots, N. \tag{11}$$

Here Δt is a fixed timestep size.



Backward-Euler Scheme

We find $(\boldsymbol{q}_h^n, u_h^n, \widehat{u}_h^n) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h^n, \boldsymbol{v})_{\mathcal{T}_h} + (u_h^n, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h^n, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\
(\frac{u_h^n - u_h^{n-1}}{\Delta t}, w)_{\mathcal{T}_h} + (\boldsymbol{q}_h^n, \nabla w)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{q}}_h^n\cdot\boldsymbol{n}, w\rangle_{\partial\mathcal{T}_h} &= (f^n, w)_{\mathcal{T}_h},\\
\langle \widehat{\boldsymbol{q}}_h^n\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \langle \widehat{u}_h^n - g_{\mathrm{D}}^n, \mu\rangle_{\partial\Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}^n, \mu\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{12}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ and for $n = 1, \ldots, N$, and

$$
(u_h^0, w)_K = (u_0, w)_{\mathcal{T}_h}, \quad \forall w \in W_h^k. \tag{13}
$$

Here the numerical flux is defined as

$$
\widehat{\boldsymbol{q}}_h^n = \boldsymbol{q}_h^n - \tau(u_h^n - \widehat{u}_h^n)\boldsymbol{n} \,. \tag{14}
$$

Backward-Euler Scheme

We find $(\boldsymbol{q}_h^n, u_h^n, \widehat{u}_h^n) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
&(\kappa^{-1}\boldsymbol{q}_h^n, \boldsymbol{v})_{\mathcal{T}_h} + (u_h^n, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} \\
&\qquad - \langle \widehat{u}_h^n, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \\
&(\frac{u_h^n}{\Delta t}, w)_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{q}_h^n, w)_{\mathcal{T}_h} \\
&\qquad + \langle \tau(u_h^n - \widehat{u}_h^n), w\rangle_{\partial\mathcal{T}_h} = (f^n, w)_{\mathcal{T}_h} + (\frac{u_h^{n-1}}{\Delta t}, w)_{\mathcal{T}_h}, \\
&\langle \boldsymbol{q}_h^n\cdot\boldsymbol{n} - \tau(u_h^n - \widehat{u}_h^n), \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} \\
&\qquad + \langle \widehat{u}_h^n, \mu\rangle_{\partial\Omega_\mathrm{D}} = \langle g_\mathrm{N}^n, \mu\rangle_{\partial\Omega_\mathrm{N}} + \langle g_\mathrm{D}^n, \mu\rangle_{\partial\Omega_\mathrm{D}},
\end{aligned}
\tag{15}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ and for $n = 1, \ldots, N$, and

$$
(u_h^0, w)_K = (u_0, w)_{\mathcal{T}_h}, \quad \forall w \in W_h^k. \tag{16}
$$

Energy Identity

Choosing $(\boldsymbol{v}, w, \mu) = (\boldsymbol{q}_h^n, u_h^n, \widehat{u}_h^n)$ and summing the equations we get

$$
\begin{aligned}
&(\frac{u_h^n - u_h^{n-1}}{\Delta t}, u_h^n)_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{q}_h^n, \boldsymbol{q}_h^n)_{\mathcal{T}_h} \\
&+\langle \tau(u_h^n - \widehat{u}_h^n), (u_h^n - \widehat{u}_h^n)\rangle_{\partial\mathcal{T}_h} = (f^n, u_h^n)_{\mathcal{T}_h} + \langle g_{\rm N}^n, \widehat{u}_h^n\rangle_{\partial\Omega_{\rm N}} + \langle g_{\rm D}^n, \widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n}\rangle_{\partial\Omega_{\rm D}} .
\end{aligned}
\tag{17}
$$

However, since

$$
(u_h^{n-1}, u_h^n)_{\mathcal{T}_h} \le \frac{1}{2}(u_h^n, u_h^n)_{\mathcal{T}_h} + \frac{1}{2}(u_h^{n-1}, u_h^{n-1})_{\mathcal{T}_h} \tag{18}
$$

we have

$$
\begin{aligned}
&(\frac{u_h^n}{2\Delta t}, u_h^n)_{\mathcal{T}_h} - (\frac{u_h^{n-1}}{2\Delta t}, u_h^{n-1})_{\mathcal{T}_h} + (\kappa^{-1}\boldsymbol{q}_h^n, \boldsymbol{q}_h^n)_{\mathcal{T}_h} \\
&+\langle \tau(u_h^n - \widehat{u}_h^n), (u_h^n - \widehat{u}_h^n)\rangle_{\partial\mathcal{T}_h} \le (f^n, u_h^n)_{\mathcal{T}_h} + \langle g_{\rm N}^n, \widehat{u}_h^n\rangle_{\partial\Omega_{\rm N}} + \langle g_{\rm D}^n, \widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n}\rangle_{\partial\Omega_{\rm D}} .
\end{aligned}
\tag{19}
$$

Energy Identity

Summing over all time steps we get

$$
\begin{aligned}
&\left(\frac{u_h^N}{2\Delta t}, u_h^N\right)_{\mathcal{T}_h} + \sum_{n=1}^{N} \left\{ (\kappa^{-1} \boldsymbol{q}_h^n, \boldsymbol{q}_h^n)_{\mathcal{T}_h} + \langle \tau (u_h^n - \widehat{u}_h^n), (u_h^n - \widehat{u}_h^n) \rangle_{\partial \mathcal{T}_h} \right\} \\
&\leq \left(\frac{u_h^0}{2\Delta t}, u_h^0\right)_{\mathcal{T}_h} + \sum_{n=1}^{N} \left\{ (f^n, u_h^n)_{\mathcal{T}_h} + \langle g_{\mathrm{N}}^n, \widehat{u}_h^n \rangle_{\partial \Omega_{\mathrm{N}}} + \langle g_{\mathrm{D}}^n, \widehat{\boldsymbol{q}}_h^n \rangle_{\partial \Omega_{\mathrm{D}}} \right\}.
\end{aligned}
\tag{20}
$$

This is the desired energy statement.

High-Order BDF Schemes

We find $(\boldsymbol{q}_h^n, u_h^n, \widehat{u}_h^n) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h^n, \boldsymbol{v})_{\mathcal{T}_h} + (u_h^n, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h^n, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0, \\
\left(\frac{\sum_{j=0}^m \alpha_j u_h^{n-j}}{\Delta t}, w\right)_{\mathcal{T}_h} + (\boldsymbol{q}_h^n, \nabla w)_{\mathcal{T}_h} \qquad & \\
- \langle \widehat{\boldsymbol{q}}_h^n\cdot\boldsymbol{n}, w\rangle_{\partial\mathcal{T}_h} &= (f^n, w)_{\mathcal{T}_h}, \\
\langle \widehat{\boldsymbol{q}}_h^n\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \langle \widehat{u}_h^n - g_{\mathrm{D}}^n, \mu\rangle_{\partial\Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}^n, \mu\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{21}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ and for $n = 1, \dots, N$, and

$$
(u_h^0, w)_K = (u_0, w)_{\mathcal{T}_h}, \quad \forall w \in W_h^k. \tag{22}
$$

Here the numerical flux is defined as

$$
\widehat{\boldsymbol{q}}_h^n = \boldsymbol{q}_h^n - \tau(u_h^n - \widehat{u}_h^n)\boldsymbol{n}\,. \tag{23}
$$

High-Order BDF Schemes

- Second-order BDF scheme

$$\alpha_0 = 3/2, \qquad \alpha_1 = -4/2, \qquad \alpha_2 = 1/2 \tag{24}$$

- Third-order BDF scheme

$$\alpha_0 = 11/6, \quad \alpha_1 = -18/6, \quad \alpha_2 = 9/6, \quad \alpha_3 = -2/6 \tag{25}$$

- Fourth-order BDF scheme

$$\begin{aligned} \alpha_0 = 25/12, \quad \alpha_1 = -48/12, \quad \alpha_2 = 36/12, \\ \alpha_3 = -16/12, \quad \alpha_4 = 3/12 \end{aligned} \tag{26}$$

High-Order BDF Schemes

We find $(\boldsymbol{q}_h^n, u_h^n, \widehat{u}_h^n) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k(g_{\mathrm{D}}^n)$ such that

$$
\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h^n, \boldsymbol{v})_{\mathcal{T}_h} + (u_h^n, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h^n, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
\left(\frac{\alpha_0 u_h^n}{\Delta t}, w \right)_{\mathcal{T}_h} + (\boldsymbol{q}_h^n, \nabla w)_{\mathcal{T}_h} & \\
- \langle \widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n}, w \rangle_{\partial \mathcal{T}_h} &= (\tilde{f}^n, w)_{\mathcal{T}_h}, \\
\langle \widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega_D} + \langle \widehat{u}_h^n - g_{\mathrm{D}}^n, \mu \rangle_{\partial \Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}^n, \mu \rangle_{\partial \Omega_{\mathrm{N}}},
\end{aligned}
\tag{27}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k(0)$ and for $n = 1, \ldots, N$, where

$$
\tilde{f}^n = f^n - \frac{\sum_{j=1}^m \alpha_j u_h^{n-j}}{\Delta t}. \tag{28}
$$

The implementation follows from that for the diffusion equation.

DIRK Schemes: General Formula

Let us review the general RK formula for the ODE

$$y' = r(y). \tag{29}$$

The s-stage RK method for the ODE is defined as

$$y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i r(y_{n,i}) \tag{30}$$

where

$$y_{n,i} = y_n + \Delta t \sum_{j=1}^{s} a_{ij} r(y_{n,j}), \quad i = 1, \ldots, s. \tag{31}$$

Here s is the number of stages, b_i are the weights, a_{ij} are internal coefficients, and $y_{n,i}$ is an approximation to $y(t)$ at $t = (t_n + c_i \Delta t)$.

DIRK Schemes: Butcher's Table

The coefficients of s-stage DIRK schemes are usually given in the form of the Butcher's Table:

$$\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{c} \\ \hline \boldsymbol{b} & \end{array} \quad \equiv \quad \begin{array}{cccc|c} a_{11} & 0 & \dots & 0 & c_1 \\ a_{21} & a_{22} & \dots & 0 & c_2 \\ \vdots & & \dots & \vdots & \vdots \\ a_{s1} & a_{s2} & \dots & a_{ss} & c_s \\ \hline b_1 & b_2 & \dots & b_s & \end{array} \tag{32}$$

Here the matrix $\boldsymbol{A}$ is **lower-triangular** and **invertible**.

DIRK Schemes: Reformulation

Since the matrix $\boldsymbol{A}$ is invertible we denote its inverse by $\boldsymbol{D}$. We can reformulate the RK formula (30)-(31) as follows:

$$\sum_{j=1}^{s} d_{ij} \left(\frac{y_{n,j} - y_n}{\Delta t} \right) = r(y_{n,i}), \quad i = 1, \ldots, s, \tag{33}$$

and

$$y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i \sum_{j=1}^{s} d_{ij} \left(\frac{y_{n,j} - y_n}{\Delta t} \right). \tag{34}$$

DIRK Schemes: Reformulation

The coefficients of s-stage reformulated DIRK schemes are also given in the form of the Butcher's Table:

$$\begin{array}{c|c} \boldsymbol{D} & \boldsymbol{c} \\ \hline \boldsymbol{b} & \end{array} \quad \equiv \quad \begin{array}{cccc|c} d_{11} & 0 & \dots & 0 & c_1 \\ d_{21} & d_{22} & \dots & 0 & c_2 \\ \vdots & & \dots & \vdots & \vdots \\ d_{s1} & d_{s2} & \dots & d_{ss} & c_s \\ \hline b_1 & b_2 & \dots & b_s & \end{array} \tag{35}$$

Here the matrix $\boldsymbol{D}$ is also a **lower-triangular** matrix like $\boldsymbol{A}$.

DIRK Schemes: Reformulation

As a result, we obtain a uncoupled system

$$
\begin{aligned}
d_{11}\left(\frac{y_{n,1}-y_n}{\Delta t}\right) &= r(y_{n,1}),\\
d_{22}\left(\frac{y_{n,2}-y_n}{\Delta t}\right) + d_{21}\left(\frac{y_{n,1}-y_n}{\Delta t}\right) &= r(y_{n,2}),\\
&\cdots\\
d_{ss}\left(\frac{y_{n,s}-y_n}{\Delta t}\right) + \sum_{j=1}^{s-1} d_{sj}\left(\frac{y_{n,j}-y_n}{\Delta t}\right) &= r(y_{n,s}),
\end{aligned} \tag{36}
$$

and

$$
y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i \sum_{j=1}^{s} d_{ij}\left(\frac{y_{n,j}-y_n}{\Delta t}\right). \tag{37}
$$

We shall adopt this reformulated RK formula for the HDG method.

HDG-DIRK Methods

First, we find $(\boldsymbol{q}_h^{n,1}, u_h^{n,1}, \widehat{u}_h^{n,1}) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
&(\kappa^{-1}\boldsymbol{q}_h^{n,1}, \boldsymbol{v})_{\mathcal{T}_h} + (u_h^{n,1}, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \left\langle \widehat{u}_h^{n,1}, \boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} = 0,\\
&\left(d_{11}\frac{u_h^{n,1} - u_h^n}{\Delta t}, w\right)_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{q}_h^{n,1}, w)_{\mathcal{T}_h}\\
&\qquad + \left\langle \tau(u_h^{n,1} - \widehat{u}_h^{n,1}), w\right\rangle_{\partial\mathcal{T}_h} = (f^{n,1}, w)_{\mathcal{T}_h},\\
&\left\langle \boldsymbol{q}_h^{n,1}\cdot\boldsymbol{n} - \tau(u_h^{n,1} - \widehat{u}_h^{n,1}), \mu\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D}\\
&\qquad + \left\langle \widehat{u}_h^{n,1} - g_{\mathrm{D}}^{n,1}, \mu\right\rangle_{\partial\Omega_{\mathrm{D}}} = \left\langle g_{\mathrm{N}}^{n,1}, \mu\right\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{38}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$.

Then we find $(\boldsymbol{q}_h^{n,2}, u_h^{n,2}, \widehat{u}_h^{n,2}) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$(\kappa^{-1}\boldsymbol{q}_h^{n,2}, \boldsymbol{v})_{\mathcal{T}_h} + (u_h^{n,2}, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \left\langle \widehat{u}_h^{n,2}, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} = 0,$$

$$\left(d_{22}\frac{u_h^{n,2} - u_h^n}{\Delta t}, w\right)_{\mathcal{T}_h} + \left(d_{21}\frac{u_h^{n,1} - u_h^n}{\Delta t}, w\right)_{\mathcal{T}_h} - (\nabla \cdot \boldsymbol{q}_h^{n,2}, w)_{\mathcal{T}_h} + \left\langle \tau(u_h^{n,2} - \widehat{u}_h^{n,2}), w \right\rangle_{\partial \mathcal{T}_h} = (f^{n,2}, w)_{\mathcal{T}_h},$$

$$\left\langle \boldsymbol{q}_h^{n,2} \cdot \boldsymbol{n} - \tau(u_h^{n,2} - \widehat{u}_h^{n,2}), \mu \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega_D} + \left\langle \widehat{u}_h^{n,2} - g_{\mathrm{D}}^{n,2}, \mu \right\rangle_{\partial \Omega_{\mathrm{D}}} = \left\langle g_{\mathrm{N}}^{n,2}, \mu \right\rangle_{\partial \Omega_{\mathrm{N}}}, \tag{39}$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$.

HDG-DIRK Methods

So on, we find $(\boldsymbol{q}_h^{n,s}, u_h^{n,s}, \widehat{u}_h^{n,s}) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
&(\kappa^{-1}\boldsymbol{q}_h^{n,s}, \boldsymbol{v})_{\mathcal{T}_h} + (u_h^{n,s}, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h^{n,s}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0,\\
&\left(d_{ss}\frac{u_h^{n,s}-u_h^n}{\Delta t}, w\right)_{\mathcal{T}_h} + \left(\sum_{j=1}^{s-1} d_{sj}\frac{u_h^{n,j}-u_h^n}{\Delta t}, w\right)_{\mathcal{T}_h}\\
&\qquad -(\nabla\cdot\boldsymbol{q}_h^{n,s}, w)_{\mathcal{T}_h} + \langle \tau(u_h^{n,s}-\widehat{u}_h^{n,s}), w\rangle_{\partial\mathcal{T}_h} = (f^{n,s}, w)_{\mathcal{T}_h},\\
&\langle \boldsymbol{q}_h^{n,s}\cdot\boldsymbol{n} - \tau(u_h^{n,s}-\widehat{u}_h^{n,s}), \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D}\\
&\qquad + \langle \widehat{u}_h^{n,s} - g_{\mathrm{D}}^{n,s}, \mu\rangle_{\partial\Omega_{\mathrm{D}}} = \langle g_{\mathrm{N}}^{n,s}, \mu\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{40}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$.

Finally, we determine

$$u_h^{n+1} = u_h^n + \Delta t \sum_{i=1}^{s} b_i \sum_{j=1}^{s} d_{ij} \left(\frac{u_h^{n,j} - u_h^n}{\Delta t} \right), \tag{41}$$

and find $(\boldsymbol{q}_h^{n+1}, \widehat{u}_h^{n+1}) \in \boldsymbol{V}_h^k \times M_h^k$ such that

$$\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h^{n+1}, \boldsymbol{v})_{\mathcal{T}_h} - \left\langle \widehat{u}_h^{n+1}, \boldsymbol{v}\cdot\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} + (u_h^{n+1}, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} &= 0, \\
\left\langle \boldsymbol{q}_h^{n+1}\cdot\boldsymbol{n} - \tau(u_h^{n+1} - \widehat{u}_h^{n+1}), \mu \right\rangle_{\partial\mathcal{T}_h \backslash \partial\Omega_D} & \\
+ \left\langle \widehat{u}_h^{n+1} - g_D^{n+1}, \mu \right\rangle_{\partial\Omega_\mathrm{D}} &= \left\langle g_\mathrm{N}^{n+1}, \mu \right\rangle_{\partial\Omega_\mathrm{N}},
\end{aligned} \tag{42}$$

for all $(\boldsymbol{v}, \mu) \in \boldsymbol{V}_h^k \times M_h^k$.

Note that L-stable DIRK methods do not require us to compute (41) and (41).

Local Postprocessing

Find $u_h^{n\,*} \in \mathcal{P}_{k+1}(K)$ such that on every $K \in \mathcal{T}_h$,

$$\begin{aligned} \left(\kappa \nabla u_h^{n\,*}, \nabla w\right)_K &= \left(\boldsymbol{q}_h^{n\,*}, \nabla w\right)_K, \quad \forall w \in \mathcal{P}_{k+1}(K), \\ \left(u_h^{n\,*}, 1\right)_K &= \left(u_h^{n}, 1\right)_K. \end{aligned} \tag{43}$$

This system is very inexpensive to compute.

The local postprocessing can be performed at **any time steps** where an enhanced accuracy in the solution is needed.

Convergence Properties

Let m be the order of accuracy of temporal discretization. We have

$$\|u - u_h\|_{L^2(\mathcal{T}_h)} \leq C|u|_{H^{k+1}(\mathcal{T}_h)} h^{\min(k+1,m)}, \tag{44}$$

and that

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\mathcal{T}_h)} \leq C|\boldsymbol{q}|_{H^{k+1}(\mathcal{T}_h)} h^{\min(k+1,m)}, \tag{45}$$

and that

$$\|u - u_h^*\|_{L^2(\mathcal{T}_h)} \leq C|u|_{H^{k+2}(\mathcal{T}_h)} h^{\min(k+2,m)}. \tag{46}$$

The order of convergence depends on both k and m.

Wave Equation

We consider the following wave equation

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\kappa \nabla u) &= f, && \text{in } \Omega \times (0, T], \\
u &= u_{\mathrm{D}}, && \text{on } \partial\Omega_{\mathrm{D}} \times (0, T], \\
\kappa \nabla u \cdot \boldsymbol{n} &= g_{\mathrm{N}}, && \text{on } \partial\Omega_{\mathrm{N}} \times (0, T], \\
u &= u_0, && \text{on } \Omega \times \{0\}, \\
\frac{\partial u}{\partial t} &= v_0, && \text{on } \Omega \times \{0\}.
\end{aligned}
\tag{47}
$$

Here κ is the propagation speed of the wave.

Note that the heat equation is parabolic, while the wave equation is hyperbolic.

We introduce new variables

$$v = \frac{\partial u}{\partial t}, \qquad \boldsymbol{q} = \kappa \nabla u, \qquad \boldsymbol{p} = \kappa \nabla v \tag{48}$$

and write the wave equation as

$$\begin{aligned}
\frac{\partial \boldsymbol{q}}{\partial t} - \kappa \nabla v &= 0, && \text{in } \Omega \times (0, T], \\
\frac{\partial v}{\partial t} - \nabla \cdot \boldsymbol{q} &= f, && \text{in } \Omega \times (0, T], \\
v &= g_{\mathrm{D}}, && \text{on } \partial\Omega_{\mathrm{D}} \times (0, T], \\
\boldsymbol{q} \cdot \boldsymbol{n} &= g_{\mathrm{N}}, && \text{on } \partial\Omega_{\mathrm{N}}, \times (0, T], \\
v &= v_0, && \text{on } \Omega \times \{0\}, \\
\boldsymbol{q} &= \boldsymbol{q}_0, && \text{on } \Omega \times \{0\}.
\end{aligned} \tag{49}$$

Semi-Discrete HDG Formulation

We find $(\boldsymbol{q}_h, v_h, \widehat{v}_h) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\frac{\partial \boldsymbol{q}_h}{\partial t}, \boldsymbol{v})_{\mathcal{T}_h} + (v_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{v}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\
(\frac{\partial v_h}{\partial t}, w)_{\mathcal{T}_h} + (\boldsymbol{q}_h, \nabla w)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial\mathcal{T}_h} &= (f, w)_{\mathcal{T}_h},\\
\langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \langle \widehat{v}_h - g_{\mathrm{D}}, \mu\rangle_{\partial\Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}, \mu\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{50}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ and for all $t \in (0, T)$, and that

$$
\begin{aligned}
(v_h(t=0), w)_K &= (v_0, w)_{\mathcal{T}_h}, \quad \forall w \in W_h^k,\\
(\boldsymbol{q}_h(t=0), \boldsymbol{v})_K &= (\boldsymbol{q}_0, \boldsymbol{v})_{\mathcal{T}_h}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h^k,
\end{aligned}
\tag{51}
$$

where

$$
\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h - \tau(v_h - \widehat{v}_h)\boldsymbol{n}\,. \tag{52}
$$

This completes the semi-discrete formulation.

HDG-BDF Methods: Formulation

We find $(\boldsymbol{q}_h^n, v_h^n, \widehat{v}_h^n) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
&(\kappa^{-1} \frac{\sum_{j=0}^{m} \alpha_j \boldsymbol{q}_h^{n-j}}{\Delta t}, \boldsymbol{v})_{\mathcal{T}_h} + (v_h^n, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{v}_h^n, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0, \\
&(\frac{\sum_{j=0}^{m} \alpha_j v_h^{n-j}}{\Delta t}, w)_{\mathcal{T}_h} + (\boldsymbol{q}_h^n, \nabla w)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n}, w \rangle_{\partial \mathcal{T}_h} = (f^n, w)_{\mathcal{T}_h}, \\
&\langle \widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega_D} + \langle \widehat{v}_h^n - g_{\mathrm{D}}^n, \mu \rangle_{\partial \Omega_{\mathrm{D}}} = \langle g_{\mathrm{N}}^n, \mu \rangle_{\partial \Omega_{\mathrm{N}}},
\end{aligned} \tag{53}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ and that

$$
\begin{aligned}
(v_h(t=0), w)_K &= (v_0, w)_{\mathcal{T}_h}, \quad \forall w \in W_h^k, \\
(\boldsymbol{q}_h(t=0), \boldsymbol{v})_K &= (\boldsymbol{q}_0, \boldsymbol{v})_{\mathcal{T}_h}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h^k,
\end{aligned} \tag{54}
$$

where

$$
\widehat{\boldsymbol{q}}_h^n = \boldsymbol{q}_h^n - \tau (v_h^n - \widehat{v}_h^n) \boldsymbol{n} . \tag{55}
$$

HDG-BDF Methods: Formulation

We find $(\boldsymbol{q}_h^n, v_h^n, \widehat{v}_h^n) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\frac{\alpha_0 \boldsymbol{q}_h^n}{\Delta t}, \boldsymbol{v})_{\mathcal{T}_h} + (v_h^n, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{v}_h^n, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= (\tilde{\boldsymbol{y}}^n, \boldsymbol{v})_{\mathcal{T}_h}, \\
(\frac{\alpha_0 v_h^n}{\Delta t}, w)_{\mathcal{T}_h} + (\boldsymbol{q}_h^n, \nabla w)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{q}}_h^n\cdot\boldsymbol{n}, w\rangle_{\partial\mathcal{T}_h} &= (\tilde{f}^n, w)_{\mathcal{T}_h}, \\
\langle \widehat{\boldsymbol{q}}_h^n\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \langle \widehat{v}_h^n - g_{\mathrm{D}}^n, \mu\rangle_{\partial\Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}^n, \mu\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{56}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$, where

$$
\tilde{\boldsymbol{y}}^n = -\kappa^{-1}\frac{\sum_{j=1}^m \alpha_j \boldsymbol{q}_h^{n-j}}{\Delta t}, \qquad \tilde{f}^n = f^n - \frac{\sum_{j=1}^m \alpha_j v_h^{n-j}}{\Delta t}. \tag{57}
$$

HDG-BDF Methods: Formulation

Once v_h^n is already computed, we can compute $u_h^n \in \mathcal{P}_k(K)$ by **locally solving**

$$\left(\frac{\alpha_0 u_h^n}{\Delta t}, w\right)_K = (v_h^n, w)_K - \left(\frac{\sum_{j=1}^m \alpha_j u_h^{n-j}}{\Delta t}, w\right)_K, \quad w \in \mathcal{P}_k(K). \tag{58}$$

We can also compute $\boldsymbol{p}_h^n \in [\mathcal{P}_k(K)]^d$ such that

$$\left(\kappa^{-1}\boldsymbol{p}_h^n, \boldsymbol{v}\right)_K = \langle \widehat{v}_h^n, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_K - (v_h^n, \nabla\cdot\boldsymbol{v})_K, \quad \forall \boldsymbol{v} \in [\mathcal{P}_k(K)]^d. \tag{59}$$

In summary, we already compute $(\boldsymbol{q}_h^n, v_h^n, \widehat{v}_h^n)$ and $(\boldsymbol{p}_h^n, u_h^n)$.

HDG-DIRK Methods: Formulation

First we find $(\boldsymbol{q}_h^{n,1}, v_h^{n,1}, \widehat{v}_h^{n,1}) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\frac{d_{11}\boldsymbol{q}_h^{n,1}}{\Delta t}, \boldsymbol{v})_{\mathcal{T}_h} + (v_h^{n,1}, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \left\langle \widehat{v}_h^{n,1}, \boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} &= (\tilde{\boldsymbol{y}}^{n,1}, \boldsymbol{v})_{\mathcal{T}_h}, \\
(\frac{d_{11}v_h^{n,1}}{\Delta t}, w)_{\mathcal{T}_h} + (\boldsymbol{q}_h^{n,1}, \nabla w)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{q}}_h^{n,1}\cdot\boldsymbol{n}, w\right\rangle_{\partial\mathcal{T}_h} &= (\tilde{f}^{n,1}, w)_{\mathcal{T}_h}, \\
\left\langle \widehat{\boldsymbol{q}}_h^{n,1}\cdot\boldsymbol{n}, \mu\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \left\langle \widehat{v}_h^{n,1} - g_{\mathrm{D}}^{n,1}, \mu\right\rangle_{\partial\Omega_{\mathrm{D}}} &= \left\langle g_{\mathrm{N}}^{n,1}, \mu\right\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{60}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$, where

$$
\tilde{\boldsymbol{y}}^{n,1} = \kappa^{-1}\frac{d_{11}}{\Delta t}\boldsymbol{q}_h^n, \qquad \tilde{f}^{n,1} = f^n + \frac{d_{11}}{\Delta t}v_h^n. \tag{61}
$$

HDG-DIRK Methods: Formulation

Next, we find $(\boldsymbol{q}_h^{n,2}, v_h^{n,2}, \widehat{v}_h^{n,2}) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\frac{d_{22}\boldsymbol{q}_h^{n,2}}{\Delta t}, \boldsymbol{v})_{\mathcal{T}_h} + (v_h^{n,2}, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \left\langle \widehat{v}_h^{n,2}, \boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} &= (\tilde{\boldsymbol{y}}^{n,2}, \boldsymbol{v})_{\mathcal{T}_h}, \\
(\frac{d_{22}v_h^{n,2}}{\Delta t}, w)_{\mathcal{T}_h} + (\boldsymbol{q}_h^{n,2}, \nabla w)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{q}}_h^{n,2}\cdot\boldsymbol{n}, w\right\rangle_{\partial\mathcal{T}_h} &= (\tilde{f}^{n,2}, w)_{\mathcal{T}_h}, \\
\left\langle \widehat{\boldsymbol{q}}_h^{n,2}\cdot\boldsymbol{n}, \mu\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \left\langle \widehat{v}_h^{n,2} - g_{\mathrm{D}}^{n,2}, \mu\right\rangle_{\partial\Omega_{\mathrm{D}}} &= \left\langle g_{\mathrm{N}}^{n,2}, \mu\right\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{62}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$, where

$$
\begin{aligned}
\tilde{\boldsymbol{y}}^{n,2} &= \kappa^{-1}\frac{d_{22}}{\Delta t}\boldsymbol{q}_h^n - \kappa^{-1}\frac{d_{21}}{\Delta t}(\boldsymbol{q}_h^{n,1} - \boldsymbol{q}_h^n), \\
\tilde{f}^{n,2} &= f^n + \frac{d_{22}}{\Delta t}v_h^n - \frac{d_{21}}{\Delta t}(v_h^{n,1} - v_h^n).
\end{aligned}
\tag{63}
$$

HDG-DIRK Methods: Formulation

So on, we find $(\boldsymbol{q}_h^{n,s}, v_h^{n,s}, \widehat{v}_h^{n,s}) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\frac{d_{ss}\boldsymbol{q}_h^{n,s}}{\Delta t}, \boldsymbol{v})_{\mathcal{T}_h} + (v_h^{n,s}, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{v}_h^{n,s}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= (\tilde{\boldsymbol{y}}^{n,s}, \boldsymbol{v})_{\mathcal{T}_h}, \\
(\frac{d_{ss}v_h^{n,s}}{\Delta t}, w)_{\mathcal{T}_h} + (\boldsymbol{q}_h^{n,s}, \nabla w)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{q}}_h^{n,s}\cdot\boldsymbol{n}, w\rangle_{\partial\mathcal{T}_h} &= (\tilde{f}^{n,s}, w)_{\mathcal{T}_h}, \\
\langle \widehat{\boldsymbol{q}}_h^{n,s}\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \langle \widehat{v}_h^{n,s} - g_{\mathrm{D}}^{n,s}, \mu\rangle_{\partial\Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}^{n,s}, \mu\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{64}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$, where

$$
\begin{aligned}
\tilde{\boldsymbol{y}}^{n,s} &= \kappa^{-1}\frac{d_{ss}}{\Delta t}\boldsymbol{q}_h^n - \kappa^{-1}\sum_{j=1}^{s-1}\frac{d_{sj}}{\Delta t}(\boldsymbol{q}_h^{n,j} - \boldsymbol{q}_h^n), \\
\tilde{f}^{n,s} &= f^n + \frac{d_{ss}}{\Delta t}v_h^n - \sum_{j=1}^{s-1}\frac{d_{sj}}{\Delta t}(v_h^{n,j} - v_h^n).
\end{aligned}
\tag{65}
$$

Finally, we determine

$$v_h^{n+1} = v_h^n + \Delta t \sum_{i=1}^{s} b_i \sum_{j=1}^{s} d_{ij} \left(\frac{v_h^{n,j} - v_h^n}{\Delta t} \right), \tag{66}$$

and find $(\boldsymbol{q}_h^{n+1}, \widehat{v}_h^{n+1}) \in \boldsymbol{V}_h^k \times M_h^k$ such that

$$\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h^{n+1}, \boldsymbol{v})_{\mathcal{T}_h} - \left\langle \widehat{v}_h^{n+1}, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} + (v_h^{n+1}, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} &= 0, \\
\left\langle \boldsymbol{q}_h^{n+1} \cdot \boldsymbol{n} - \tau (v_h^{n+1} - \widehat{v}_h^{n+1}), \mu \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega_D} & \\
+ \left\langle \widehat{v}_h^{n+1} - g_D^{n+1}, \mu \right\rangle_{\partial \Omega_{\mathrm{D}}} &= \left\langle g_{\mathrm{N}}^{n+1}, \mu \right\rangle_{\partial \Omega_{\mathrm{N}}},
\end{aligned} \tag{67}$$

for all $(\boldsymbol{v}, \mu) \in \boldsymbol{V}_h^k \times M_h^k$.

Note that L-stable DIRK methods do not require us to compute (66) and (67).

HDG-DIRK Methods: Formulation

We can also compute $\boldsymbol{p}_h^{n+1} \in [\mathcal{P}_k(K)]^d$ such that

$$\left(\kappa^{-1}\boldsymbol{p}_h^{n+1}, \boldsymbol{v}\right)_K = \left\langle \widehat{v}_h^{n+1}, \boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_K - (v_h^{n+1}, \nabla\cdot\boldsymbol{v})_K, \quad \forall \boldsymbol{v} \in [\mathcal{P}_k(K)]^d. \tag{68}$$

To compute u_h^{n+1}, we apply DIRK methods to solve

$$\left(\frac{\partial u_h}{\partial t}, w\right)_K = (v_h, w)_K\,. \tag{69}$$

In summary, we already compute $(\boldsymbol{q}_h^n, v_h^n, \widehat{v}_h^n)$ and $(\boldsymbol{p}_h^n, u_h^n)$ for all time steps.

Local Postprocessing

We compute $u_h^{n\,*} \in \mathcal{P}_{k+1}(K)$ such that

$$\begin{aligned} (\kappa \nabla u_h^{n\,*}, \nabla w)_K &= (\boldsymbol{q}_h^n, \nabla w)_K\,, \quad \forall w \in \mathcal{P}_{k+1}(K), \\ (u_h^{n\,*}, 1)_K &= (u_h^n, 1)_K\,, \end{aligned} \tag{70}$$

and $v_h^{n\,*} \in \mathcal{P}_{k+1}(K)$ such that

$$\begin{aligned} (\kappa \nabla v_h^{n\,*}, \nabla w)_K &= (\boldsymbol{p}_h^n, \nabla w)_K\,, \quad \forall w \in \mathcal{P}_{k+1}(K), \\ (v_h^{n\,*}, 1)_K &= (v_h^n, 1)_K\,. \end{aligned} \tag{71}$$

This postprocessing is inexpensive.

The local postprocessing can be performed at **any time steps** where an enhanced accuracy in the solution is needed.

Convergence Properties

Let m be the order of accuracy of temporal discretization. We have

$$\begin{aligned}
\|u - u_h\|_{L^2(\mathcal{T}_h)} &\leq Ch^{\min(k+1,m)}, \\
\|v - v_h\|_{L^2(\mathcal{T}_h)} &\leq Ch^{\min(k+1,m)}, \\
\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\mathcal{T}_h)} &\leq Ch^{\min(k+1,m)}, \\
\|\boldsymbol{p} - \boldsymbol{p}_h\|_{L^2(\mathcal{T}_h)} &\leq Ch^{\min(k+1,m)}, \\
\|u - u_h^*\|_{L^2(\mathcal{T}_h)} &\leq Ch^{\min(k+2,m)}, \\
\|v - v_h^*\|_{L^2(\mathcal{T}_h)} &\leq Ch^{\min(k+2,m)}.
\end{aligned} \tag{72}$$

The order of convergence depends on both k and m.

References

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