HDG Method for Compressible Flow

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July 12, 2017

Outline

Today lecture's main goals:

- The Euler Equations
- The Navier-Stokes Equations

The Euler Equations for Compressible Flow

Consider the steady-state compressible Euler system

$$\nabla \cdot \boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{s}(\boldsymbol{u}), \tag{1}$$

where

$$\boldsymbol{u} = \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho E \end{pmatrix}, \quad \boldsymbol{F}(\boldsymbol{u}) = \begin{pmatrix} \rho v_1 & \rho v_2 & \rho v_3 \\ \rho v_1^2 + p & \rho v_1 v_2 & \rho v_1 v_3 \\ \rho v_1 v_2 & \rho v_2^2 + p & \rho v_2 v_3 \\ \rho v_1 v_3 & \rho v_2 v_3 & \rho v_3^2 + p \\ \rho v_1 H & \rho v_2 H & \rho v_3 H \end{pmatrix} \tag{2}$$

- Density ρ, velocity vector $\boldsymbol{v} = (v_1, v_2, v_3)$, and total energy E
- Static pressure $p = (\gamma - 1)(\rho E - 0.5\boldsymbol{v}^T\boldsymbol{v})$, with $\gamma = 1.4$ for air
- Total enthalpy $H = E + p/\rho$

The Local Solver

The HDG method looks for $\boldsymbol{u}_h \in \boldsymbol{W}_h^k$ such that

$$-\left(\boldsymbol{F}(\boldsymbol{u}_h), \nabla \boldsymbol{w}\right)_K + \left\langle \widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}), \boldsymbol{w} \right\rangle_{\partial K} = \left(\boldsymbol{s}(\boldsymbol{u}_h), \boldsymbol{w}\right)_K, \qquad (3)$$

for all $\boldsymbol{w} \in [\mathcal{P}_k(K)]^m$ and for all $K \in \mathcal{T}_h$. Here $\boldsymbol{u}_h$ is the numerical solution and $\widehat{\boldsymbol{u}}_h$ is the numerical trace.

To complete the method, we still need to define the numerical flux $\widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n})$ and the global problem for the numerical trace.

The Numerical Flux

The numerical flux is chosen as follows

$$\widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}) = \boldsymbol{F}(\widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} + \boldsymbol{S}(\widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \tag{4}$$

Here $\boldsymbol{S}(\widehat{\boldsymbol{u}}_h)$ is the stabilization tensor.

Matrix numerical flux:

$$\boldsymbol{S}(\widehat{\boldsymbol{u}}_h) = \boldsymbol{R}(\widehat{\boldsymbol{u}}_h)|\boldsymbol{\Lambda}(\widehat{\boldsymbol{u}}_h)|\boldsymbol{R}^{-1}(\widehat{\boldsymbol{u}}_h). \tag{5}$$

Here $\boldsymbol{\Lambda}(\widehat{\boldsymbol{u}}_h)$ and $\boldsymbol{R}(\widehat{\boldsymbol{u}}_h)$ are the eigenvalues and eigenvectors of the Jacobian matrix.

Lax-Friedrich's numerical flux:

$$\boldsymbol{S}(\widehat{\boldsymbol{u}}_h) = \tau \boldsymbol{I}. \tag{6}$$

Here τ is an upper bound of the absolute value of the largest magnitude eigenvalue.

The Global Problem

The HDG method looks for $\widehat{\boldsymbol{u}}_h \in \boldsymbol{M}_h^k$ such that

$$\left\langle \widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}), \boldsymbol{\mu} \right\rangle_{\partial\mathcal{T}_h \backslash \partial\Omega} + \left\langle \widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}), \boldsymbol{\mu} \right\rangle_{\partial\Omega} = 0, \qquad (7)$$

for all $\boldsymbol{\mu} \in \boldsymbol{M}_h^k$. Here $\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n})$ is the boundary flux that incorporate the boundary conditions and $\boldsymbol{u}_{bc}$ is the boundary data.

The HDG Method for the Euler Equations

The HDG method looks for $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h^k \times \boldsymbol{M}_h^k$ such that

$$
\begin{aligned}
-\left(\boldsymbol{F}(\boldsymbol{u}_h), \nabla \boldsymbol{w}\right)_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}), \boldsymbol{w} \right\rangle_{\partial \mathcal{T}_h} - \left(\boldsymbol{s}(\boldsymbol{u}_h), \boldsymbol{w}\right)_{\mathcal{T}_h} &= 0, \\
\left\langle \widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}), \boldsymbol{\mu} \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \left\langle \widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}), \boldsymbol{\mu} \right\rangle_{\partial \Omega} &= 0,
\end{aligned}
\tag{8}
$$

for all $(\boldsymbol{w}, \boldsymbol{\mu}) \in \boldsymbol{W}_h^k \times \boldsymbol{M}_h^k$, where

$$
\widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}) = \boldsymbol{F}(\widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n} + \boldsymbol{S}(\widehat{\boldsymbol{u}}_h)(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h). \tag{9}
$$

We still need to define the boundary flux $\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n})$.

Boundary Conditions

We consider two boundary conditions:

- Far field condition: The number of quantities being set to the flow freestream values is equal to the number of negative eigenvalues.
- Inviscid wall condition: Set the normal velocity to zero and extrapolate the other quantities (density, tangential velocity, and total energy).

The Boundary Flux for the Far Field Condition

The boundary flux is defined as

$$\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}) = \boldsymbol{A}_h^+(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}_h) + \boldsymbol{A}_h^-(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}_{bc}), \tag{10}$$

where $\boldsymbol{A}_h^+ = \boldsymbol{R}(|\boldsymbol{\Lambda}| + \boldsymbol{\Lambda})\boldsymbol{R}^{-1}$ and $\boldsymbol{A}_h^- = \boldsymbol{R}(|\boldsymbol{\Lambda}| - \boldsymbol{\Lambda})\boldsymbol{R}^{-1}$.

- What if all the eigenvalues are positive?
- What if all the eigenvalues are negative?
- What if some eigenvalues are positive and the other eigenvalues are negative?

The Boundary Flux for the Inviscid Wall Condition

The boundary flux is defined as

$$\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}) = \widehat{\boldsymbol{u}}_h - \widetilde{\boldsymbol{u}}_h(\boldsymbol{u}_h), \tag{11}$$

where

$$\widetilde{\boldsymbol{u}}_h(\boldsymbol{u}_h) = (\rho_h, (\rho_h \boldsymbol{v}_h)_t, \rho_h E_h)^T \tag{12}$$

Here $(\rho_h \boldsymbol{v}_h)_t = \rho_h \boldsymbol{v}_h - (\rho_h \boldsymbol{v}_h \cdot \boldsymbol{n})\boldsymbol{n}$ is the tangential component of $\rho_h \boldsymbol{v}_h$.

The Compressible Navier-Stokes Equations

We consider the steady-state compressible NS equations

$$\nabla \cdot \boldsymbol{F}(\boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{s}, \quad \text{in } \Omega, \tag{13}$$

where

$$\boldsymbol{u} = \begin{pmatrix} \rho \\ \rho v_j \\ \rho E \end{pmatrix}, \qquad \boldsymbol{F}_j = \begin{pmatrix} \rho v_j \\ \rho v_i v_j + \delta_{ij} p - \tau_{ij} \\ v_j(\rho E + p) - v_i \tau_{ij} - \kappa \frac{\partial T_j}{\partial x_j} \end{pmatrix}, \tag{14}$$

and

$$T_j = -\frac{\mu}{Pr} \frac{\partial}{\partial x_j}(E + p/\rho - v_i^2/2) \tag{15}$$

$$\tau_{ij} = \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right] \tag{16}$$

The Local Solver

The HDG method looks for $(\boldsymbol{q}_h, \boldsymbol{u}_h) \in \boldsymbol{V}_h^k \times \boldsymbol{W}_h^k$ such that

$$\begin{aligned} (\boldsymbol{q}_h, \boldsymbol{r})_K + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{r})_K - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{r} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0, \\ -\left(\boldsymbol{F}(\boldsymbol{u}_h, \boldsymbol{q}_h), \nabla \boldsymbol{w}\right)_K + \left\langle \widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}), \boldsymbol{w} \right\rangle_{\partial K} &= (\boldsymbol{s}(\boldsymbol{u}_h), \boldsymbol{w})_K, \end{aligned} \tag{17}$$

for all $(\boldsymbol{r}, \boldsymbol{w}) \in [\mathcal{P}_k(K)]^{m \times d} \times [\mathcal{P}_k(K)]^m$ and for all $K \in \mathcal{T}_h$.

To complete the method, we still need to define the numerical flux and the global problem for the numerical trace.

The Numerical Flux

The numerical flux is chosen as follows

$$\widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}) = \boldsymbol{F}(\widehat{\boldsymbol{u}}_h, \boldsymbol{q}_h)\cdot\boldsymbol{n} + (\boldsymbol{S}(\widehat{\boldsymbol{u}}_h) + \frac{\gamma}{Pr Re}\boldsymbol{I})(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \quad (18)$$

Here $\boldsymbol{S}(\widehat{\boldsymbol{u}}_h)$ is the stabilization tensor and $\boldsymbol{I}$ is the identity tensor.

Matrix numerical flux:

$$\boldsymbol{S}(\widehat{\boldsymbol{u}}_h) = \boldsymbol{R}(\widehat{\boldsymbol{u}}_h)|\boldsymbol{\Lambda}(\widehat{\boldsymbol{u}}_h)|\boldsymbol{R}^{-1}(\widehat{\boldsymbol{u}}_h). \quad (19)$$

Here $\boldsymbol{\Lambda}(\widehat{\boldsymbol{u}}_h)$ and $\boldsymbol{R}(\widehat{\boldsymbol{u}}_h)$ are the eigenvalues and eigenvectors of the Jacobian matrix.

Lax-Friedrich's numerical flux:

$$\boldsymbol{S}(\widehat{\boldsymbol{u}}_h) = \tau \boldsymbol{I}. \quad (20)$$

Here τ is an upper bound of the absolute value of the largest magnitude eigenvalue.

The Global Problem

The HDG method looks for $\widehat{\boldsymbol{u}}_h \in \boldsymbol{M}_h^k$ such that

$$\left\langle \widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}), \boldsymbol{\mu} \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \left\langle \widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}), \boldsymbol{\mu} \right\rangle_{\partial \Omega} = 0, \tag{21}$$

for all $\boldsymbol{\mu} \in \boldsymbol{M}_h^k$. Here $\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n})$ is the boundary flux that incorporate the boundary conditions and $\boldsymbol{u}_{bc}$ is the boundary data.

The HDG Method for the Navier-Stokes Equations

The HDG method find $(\boldsymbol{q}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{V}_h^k \times \boldsymbol{W}_h^k \times \boldsymbol{M}_h^k$ such that

$$
\begin{aligned}
(\boldsymbol{q}_h, \boldsymbol{r})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{r})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{r} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
-(\boldsymbol{F}(\boldsymbol{u}_h), \nabla \boldsymbol{w})_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}), \boldsymbol{w} \right\rangle_{\partial \mathcal{T}_h} - (\boldsymbol{s}(\boldsymbol{u}_h), \boldsymbol{w})_{\mathcal{T}_h} &= 0, \\
\left\langle \widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}), \boldsymbol{\mu} \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \left\langle \widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}), \boldsymbol{\mu} \right\rangle_{\partial \Omega} &= 0,
\end{aligned}
\tag{22}
$$

for all $(\boldsymbol{r}, \boldsymbol{w}, \boldsymbol{\mu}) \in \boldsymbol{V}_h^k \times \boldsymbol{W}_h^k \times \boldsymbol{M}_h^k$, where

$$
\widehat{\boldsymbol{f}_h}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}) = \boldsymbol{F}(\widehat{\boldsymbol{u}}_h, \boldsymbol{q}_h) \cdot \boldsymbol{n} + (\boldsymbol{S}(\widehat{\boldsymbol{u}}_h) + \frac{\gamma}{Pr\,Re} \boldsymbol{I})(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h). \tag{23}
$$

We still need to define the boundary flux $\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n})$.

Boundary Conditions

- Subsonic inflow condition: Set all the quantities to the boundary data, but extrapolate the total energy.
- Supersonic inflow condition: Set all the quantities to the boundary data.
- Subsonic outflow condition: Set the pressure to the given data and extrapolate the other quantities.
- Supersonic outflow condition: Extrapolate all the quantities.
- Adiabatic wall condition: Set the velocity and the last component of the numerical flux to zero and extrapolate density.
- Isothermal wall condition: Set the velocity to zero, the temperature to the given data, and extrapolate density.

The Boundary Flux for the Inflow Condition

For subsonic flow we define

$$\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}) = \widehat{\boldsymbol{u}}_h - (\rho_{bc}, \rho\boldsymbol{v}_{bc}, \rho_h E_h)^T. \tag{24}$$

For supersonic flow we define

$$\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}) = \widehat{\boldsymbol{u}}_h - \boldsymbol{u}_{bc}. \tag{25}$$

Here $\boldsymbol{u}_{bc} = (\rho_{bc}, \rho_{bc}\boldsymbol{v}_{bc}, \rho_{bc}E_{bc})$ is the data at inflow boundary.

The Boundary Flux for the Outflow Condition

For subsonic flow we define

$$\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}) = \widehat{\boldsymbol{u}}_h - \left(\rho_h, \rho_h \boldsymbol{v}_v, \frac{p_{bc}}{\gamma - 1} + 0.5|\boldsymbol{v}_h|^2\right)^T. \quad (26)$$

Here p_{bc} is the pressure data at the outflow boundary.

For supersonic flow we define

$$\widehat{\boldsymbol{b}}_h(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{bc}, \boldsymbol{n}) = \widehat{\boldsymbol{u}}_h - \boldsymbol{u}_h. \quad (27)$$

The Boundary Flux for the Wall Condition

At the solid surface with no slip condition, we define

$$\widehat{b}_{h1} = \widehat{u}_{h1} - u_{h1}, \quad \widehat{b}_{hi} = \widehat{u}_{hi}, \quad 2 \le i \le m-1. \tag{28}$$

The last component of $\widehat{\boldsymbol{b}}_h$ depends on whether the wall is isothermal or adiabatic.
For the **adiabatic wall**, we set

$$\widehat{b}_{hm} = \widehat{f}_{hm}(\boldsymbol{u}_h, \boldsymbol{q}_h, \widehat{\boldsymbol{u}}_h, \boldsymbol{n}). \tag{29}$$

For the **isothermal wall**, we set

$$\widehat{b}_{hm} = T(\widehat{\boldsymbol{u}}_h) - T_{bc}, \tag{30}$$

where T_{bc} is the wall temperature and $T(\widehat{\boldsymbol{u}}_h)$ is the numerical temperature.