# HDG Method for Convection-Diffusion

N.-C. Nguyen and J. Peraire

Massachusetts Institute of Technology
Department of Aeronautics and Astronautics

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# Outline

- HDG method for linear convection-diffusion (LCD) problems
- Local postprocessing
- Convergence properties
- Matrix structure/implementation
- Extension to nonlinear convection-diffusion problems

# LCD Problems: Problem Statement

We consider the following LCD problem

$$\begin{aligned} -\nabla \cdot (\kappa \nabla u - \boldsymbol{c} u) &= f, && \text{in } \Omega, \\ u &= g_{\mathrm{D}}, && \text{on } \partial\Omega_{\mathrm{D}}, \\ (\kappa \nabla u - \boldsymbol{c} u) \cdot \boldsymbol{n} &= g_{\mathrm{N}}, && \text{on } \partial\Omega_{\mathrm{N}}. \end{aligned} \tag{1}$$

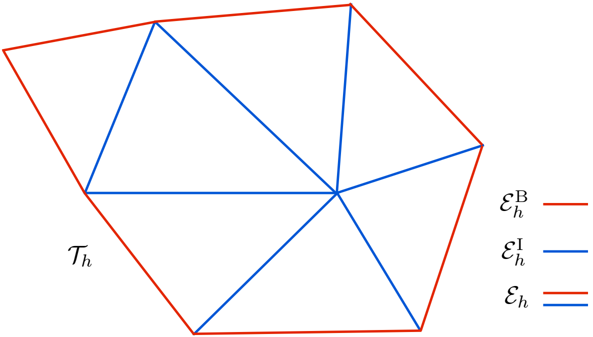
We rewrite it as

$$\begin{aligned} \boldsymbol{q} - \kappa \nabla u &= 0, && \text{in } \Omega, \\ -\nabla \cdot (\boldsymbol{q} - \boldsymbol{c} u) &= f, && \text{in } \Omega, \\ u &= g_{\mathrm{D}}, && \text{on } \partial\Omega_{\mathrm{D}}, \\ (\boldsymbol{q} - \boldsymbol{c} u) \cdot \boldsymbol{n} &= g_{\mathrm{N}}, && \text{on } \partial\Omega_{\mathrm{N}}. \end{aligned} \tag{2}$$

Here $\kappa$ is diffusion coefficient and $\boldsymbol{c}$ is convection velocity.

# LCD Problems: Finite Element Mesh

- $\mathcal{T}_h$ is the finite element triangulation of $\Omega$
- $\mathcal{E}_h^{\mathrm{B}}$ is the set of **boundary** faces
- $\mathcal{E}_h^{\mathrm{I}}$ is the set of **interior** faces
- $\mathcal{E}_h$ is the set of **all** faces.



# LCD Problems: Local Problem

Let $K$ be an element in $\mathcal{T}_h$. Let $\lambda$ be any given function on $\partial K$. We consider the following problem:

$$\begin{aligned} \boldsymbol{q}^\lambda - \kappa \nabla u^\lambda &= 0, && \text{in } K, \\ -\nabla \cdot (\boldsymbol{q}^\lambda - \boldsymbol{c} u^\lambda) &= f, && \text{in } K, \\ u^\lambda &= \lambda, && \text{on } \partial K. \end{aligned} \tag{3}$$

We observe that if

$$\lambda = u|_{\partial K}, \tag{4}$$

then

$$(\boldsymbol{q}, u) = (\boldsymbol{q}^\lambda, u^\lambda). \tag{5}$$

Hence, our goal is to find $\lambda$ that satisfies (4).

# LCD Problems: Global Problem

To that end we require that $\lambda$ satisfies

$$\lambda = g_{\mathrm{D}}, \quad \text{on } \partial\Omega_{\mathrm{D}}, \tag{6}$$

and

$$(\boldsymbol{q}^{\lambda} - \boldsymbol{c}u^{\lambda}) \cdot \boldsymbol{n} = g_{\mathrm{N}}, \quad \text{on } \partial\Omega_{\mathrm{N}}, \tag{7}$$

and

$$(\boldsymbol{q}^{\lambda} - \boldsymbol{c}u^{\lambda})^{+} \cdot \boldsymbol{n}^{+} + (\boldsymbol{q}^{\lambda} - \boldsymbol{c}u^{\lambda})^{-} \cdot \boldsymbol{n}^{-} = 0, \quad \text{on } F, \ \forall\, F \in \mathcal{E}_h^{\mathrm{I}}. \tag{8}$$

Here $\mathcal{E}_h^{\mathrm{I}}$ is the set of **interior faces**.

# LCD Problems: Abstract Formulation

We find $\lambda$ such that

$$\left\{\begin{array}{rcll} \lambda & = & g_{\mathrm{D}}, & \text{on } \partial\Omega_{\mathrm{D}}, \\ (\boldsymbol{q}^{\lambda} - \boldsymbol{c}u^{\lambda}) \cdot \boldsymbol{n} & = & g_{\mathrm{N}}, & \text{on } \partial\Omega_{\mathrm{N}}, \\ (\boldsymbol{q}^{\lambda} - \boldsymbol{c}u^{\lambda})^{+} \cdot \boldsymbol{n}^{+} + (\boldsymbol{q}^{\lambda} - \boldsymbol{c}u^{\lambda})^{-} \cdot \boldsymbol{n}^{-} & = & 0, & \text{on } F, \ \forall F \in \mathcal{E}_h^{\mathrm{I}}, \end{array}\right. \tag{9}$$

where

$$\begin{array}{rcll} \boldsymbol{q}^{\lambda} - \kappa \nabla u^{\lambda} & = & 0, & \text{in } K, \\ -\nabla \cdot (\boldsymbol{q}^{\lambda} - \boldsymbol{c}u^{\lambda}) & = & f, & \text{in } K, \\ u^{\lambda} & = & \lambda, & \text{on } \partial K. \end{array} \tag{10}$$

This abstract formulation is the **key idea** of the HDG method.

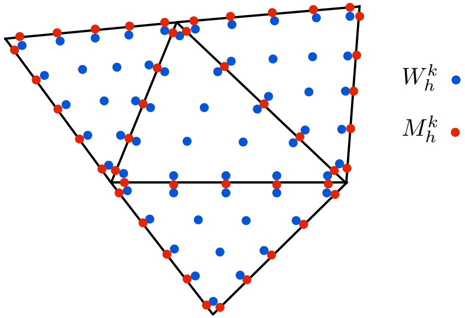
# Approximation Spaces

We introduce the following spaces

$$W_h^k = \{w \in L^2(\mathcal{T}_h) \ : \ w|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h\},$$

$$\boldsymbol{V}_h^k = \{\boldsymbol{v} \in [L^2(\mathcal{T}_h)]^d \ : \ \boldsymbol{v}|_K \in [\mathcal{P}_k(K)]^d, \forall K \in \mathcal{T}_h\},$$

$$M_h^k = \{\mu \in L^2(\mathcal{E}_h) \ : \ \mu|_F \in \mathcal{P}_k(F), \forall F \in \mathcal{E}_h\}.$$



# Inner Products

We define the volume inner products as

$$(w, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w, v)_K, \qquad (\boldsymbol{w}, \boldsymbol{v})_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{d} (w_i, v_i)_K \tag{11}$$

and the boundary inner product as

$$\langle \eta, \mu \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \langle \eta, \mu \rangle_{\partial K} \tag{12}$$

where

$$(w, v)_K = \int_K wv, \qquad \langle \eta, \mu \rangle_{\partial K} = \int_{\partial K} \eta \mu. \tag{13}$$

# Local Problem

Let $(\boldsymbol{q}_h, u_h) \in [\mathcal{P}_k(K)]^d \times \mathcal{P}_k(K)$ be such that

$$
\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h, \boldsymbol{v})_K + (u_h, \nabla \cdot \boldsymbol{v})_K - \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0, \\
(\boldsymbol{q}_h - \boldsymbol{c} u_h, \nabla w)_K - \left\langle \widehat{\boldsymbol{f}}_h \cdot \boldsymbol{n}, w \right\rangle_{\partial K} &= (f, w)_K,
\end{aligned}
\tag{14}
$$

for all $(\boldsymbol{v}, w) \in [\mathcal{P}_k(K)]^d \times \mathcal{P}_k(K)$, where

$$
\widehat{\boldsymbol{f}}_h = \boldsymbol{q}_h - \boldsymbol{c}\widehat{u}_h - \tau(u_h - \widehat{u}_h)\boldsymbol{n} \ . \tag{15}
$$

Here $\tau > 0$ is a stabilization parameter.

Note that the **total numerical flux** $\widehat{\boldsymbol{f}}_h$ includes the diffusion term, convection term, and jump term!

# Global Problem

We find $\widehat{u}_h \in M_h^k$ such that

$$\left\langle \widehat{\boldsymbol{f}}_h \cdot \boldsymbol{n}, \mu \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \langle \widehat{u}_h - g_{\mathrm{D}}, \mu \rangle_{\partial \Omega_{\mathrm{D}}} + \left\langle \widehat{\boldsymbol{f}}_h \cdot \boldsymbol{n} - g_{\mathrm{N}}, \mu \right\rangle_{\partial \Omega_{\mathrm{N}}} = 0 \quad (16)$$

for all $\mu \in M_h^k$.

The global problem (16) enforces the boundary conditions and the jump condition in the flux.

# Formulation

We find $(\boldsymbol{q}_h, u_h, \widehat{u}_h) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (u_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\
(\boldsymbol{q}_h - \boldsymbol{c}u_h, \nabla w)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{f}}_h\cdot\boldsymbol{n}, w\right\rangle_{\partial\mathcal{T}_h} &= (f, w)_{\mathcal{T}_h},\\
\left\langle \widehat{\boldsymbol{f}}_h\cdot\boldsymbol{n}, \mu\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_{\mathrm{D}}} + \langle \widehat{u}_h - g_{\mathrm{D}}, \mu\rangle_{\partial\Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}, \mu\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned}
\tag{17}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$, where

$$
\widehat{\boldsymbol{f}}_h = \boldsymbol{q}_h - \boldsymbol{c}\widehat{u}_h - \tau(u_h - \widehat{u}_h)\boldsymbol{n}\ . \tag{18}
$$

This completes the definition of the HDG method.

# Stabilization Parameter

The stabilization parameter is chosen as

$$\tau = \tau_{\text{diff}} + \tau_{\text{conv}} \tag{19}$$

where

$$\tau_{\text{diff}} = \frac{\kappa}{\ell}, \qquad \tau_{\text{conv}} = |\boldsymbol{c} \cdot \boldsymbol{n}|. \tag{20}$$

Here $\tau_{\text{diff}}$ accounts for diffusion effect, while $\tau_{\text{conv}}$ accounts for convection effect.

This choice is based on dimensional analysis that ensures all the terms in (18) have the same dimension.

# Well-posedness

**Uniqueness and existence:** Assume that $\nabla \cdot \boldsymbol{c} \geq 0$ and that $\tau$ is chosen by (19). The HDG method is well defined.

**Proof:**
We need to show that $(\boldsymbol{q}_h, u_h, \widehat{u}_h) = (\boldsymbol{0}, 0, 0)$ if $f = g_{\mathrm{D}} = g_{\mathrm{N}} = 0$. We insert (18) into (17), choose $(\boldsymbol{v}, w, \mu) = (\boldsymbol{q}_h, u_h, \widehat{u}_h)$ in (17) and sum the resulting equations up to obtain

$$\begin{aligned}(\kappa^{-1}\boldsymbol{q}_h, \boldsymbol{q}_h)_{\mathcal{T}_h} + \frac{1}{2}\left(\nabla \cdot \boldsymbol{c}, u_h^2\right)_{\mathcal{T}_h} \\ + \left\langle (\tau - \frac{1}{2}\boldsymbol{c}\cdot\boldsymbol{n})(u_h - \widehat{u}_h), (u_h - \widehat{u}_h) \right\rangle_{\partial\mathcal{T}_h} = 0.\end{aligned} \tag{21}$$

This implies that $\boldsymbol{q}_h = 0$ and $\widehat{u}_h = u_h$ on $\partial\mathcal{E}_h$ since $\nabla \cdot \boldsymbol{c} \geq 0$, and $\tau > \frac{1}{2}|\boldsymbol{c}\cdot\boldsymbol{n}|$.

## Well-posedness (cont'd)

It thus follows from the first equation of (17) that

$$\nabla u_h = 0, \tag{22}$$

which implies that $u_h$ is a constant function. Since $u_h = \widehat{u}_h = 0$ on $\partial\Omega_D$, we have $u_h = 0$. We thus obtain $\widehat{u}_h = 0$. This completes the proof.

# Expression of the Numerical Traces

It can be derived from (15) and (16) that

$$\widehat{u}_h = \begin{cases} g_{\mathrm{D}}, & \text{on } \partial\Omega_{\mathrm{D}}, \\ u_h - \dfrac{1}{\tau - \boldsymbol{c}\cdot\boldsymbol{n}}(\boldsymbol{q}_h\cdot\boldsymbol{n} - \boldsymbol{c}\cdot\boldsymbol{n}u_h - g_{\mathrm{N}}), & \text{on } \partial\Omega_{\mathrm{N}}, \\ \dfrac{\tau^+u_h^+ + \tau^-u_h^-}{\tau^+ + \tau^-} - \dfrac{1}{\tau^+ + \tau^-}(\boldsymbol{q}_h^+\cdot\boldsymbol{n}^+ + \boldsymbol{q}_h^-\cdot\boldsymbol{n}^-), & \text{on } \mathcal{E}_h^{\mathrm{I}}, \end{cases} \tag{23}$$

and that

$$\widehat{\boldsymbol{f}}_h = \begin{cases} \boldsymbol{q}_h - \boldsymbol{c}g_{\mathrm{D}} - \tau(u_h - g_{\mathrm{D}})\boldsymbol{n}, & \text{on } \partial\Omega_{\mathrm{D}} \\ g_{\mathrm{N}}\boldsymbol{n}, & \text{on } \partial\Omega_{\mathrm{N}} \\ \dfrac{\tau^+\boldsymbol{q}_h^+ + \tau^-\boldsymbol{q}_h^-}{\tau^+ + \tau^-} - \boldsymbol{c}\widehat{u}_h - \dfrac{\tau^+\tau^-}{\tau^+ + \tau^-}(u_h^+\boldsymbol{n}^+ + u_h^-\boldsymbol{n}^-), & \text{on } \mathcal{E}_h^{\mathrm{I}}, \end{cases} \tag{24}$$

# Local Postprocessing

Find $u_h^* \in \mathcal{P}_{k+1}(K)$ such that on every $K \in \mathcal{T}_h$,

$$\begin{aligned} (\kappa \nabla u_h^*, \nabla w)_K &= (\boldsymbol{q}_h, \nabla w)_K , \quad \forall w \in \mathcal{P}_{k+1}(K), \\ (u_h^*, 1)_K &= (u_h, 1)_K . \end{aligned} \tag{25}$$

This system is very inexpensive to compute.

Note that $u_h^*$ solves the following local Neumann problem

$$\begin{aligned} \nabla \cdot (\kappa \nabla u) &= \nabla \cdot \boldsymbol{q}_h, && \text{in } K, \\ \kappa \nabla u \cdot \boldsymbol{n} &= \boldsymbol{q}_h \cdot \boldsymbol{n}, && \text{on } \partial K, \\ (u, 1)_K &= (u_h, 1)_K . \end{aligned}$$

Note: *It is possible to postprocess* $\boldsymbol{q}_h$ *to obtain* $\boldsymbol{q}_h^* \in \boldsymbol{H}$**-div conforming**

# Convergence Properties

For **diffusion-dominated** problems, we have

$$\|u - u_h\|_{L^2(\mathcal{T}_h)} \leq C|u|_{H^{k+1}(\mathcal{T}_h)} h^{k+1}, \tag{26}$$

and that

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\mathcal{T}_h)} \leq C|\boldsymbol{q}|_{H^{k+1}(\mathcal{T}_h)} h^{k+1}, \tag{27}$$

and that

$$\|u - u_h^*\|_{L^2(\mathcal{T}_h)} \leq C|u|_{H^{k+2}(\mathcal{T}_h)} h^{k+2}. \tag{28}$$

Both $u_h$ and $\boldsymbol{q}_h$ converge with the **optimal order** $k+1$.
Moreover, $u_h^*$ converges with order $k+2$!

# Matrix System

The weak formulation of the HDG method yields

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} & -\mathbb{C} \\ -\mathbb{B}^T & \mathbb{D} & -\mathbb{E} \\ \mathbb{C}^T & -\mathbb{E}^T & \mathbb{M} \end{bmatrix} \begin{bmatrix} \mathbb{Q} \\ \mathbb{U} \\ \widehat{\mathbb{U}} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbb{F} \\ \mathbb{G} \end{bmatrix} \tag{29}$$

where $\mathbb{Q}$, $\mathbb{U}$ and $\widehat{\mathbb{U}}$ are the vectors of degrees of freedom of $\boldsymbol{q}_h$, $u_h$ and $\widehat{u}_h$, respectively.

Note that the matrix

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ -\mathbb{B}^T & \mathbb{D} \end{bmatrix} \tag{30}$$

is **block-diagonal** and **invertible** provided that $\tau > 0$. Its inverse can be computed inexpensively.

# Matrix System

Hence, we can eliminate $\mathbb{Q}$ and $\mathbb{U}$ to get the following system

$$\mathbb{H}\,\widehat{\mathbb{U}} = \mathbb{R}, \tag{31}$$

where

$$\begin{aligned}
\mathbb{H} &= \mathbb{M} + [\mathbb{C}^T \quad -\mathbb{E}^T] \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ -\mathbb{B}^T & \mathbb{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{C} \\ \mathbb{E} \end{bmatrix} \\
\mathbb{R} &= \mathbb{G} - [\mathbb{C}^T \quad -\mathbb{E}^T] \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ -\mathbb{B}^T & \mathbb{D} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \mathbb{F} \end{bmatrix}
\end{aligned} \tag{32}$$

The final matrix system of the HDG method involves only the degrees of freedom of $\widehat{u}_h$.

# Nonlinear Convection-Diffusion Equations

We consider the following NCD problem

$$\begin{aligned} -\nabla \cdot (\kappa \nabla u - \boldsymbol{c}(u)) &= f, && \text{in } \Omega, \\ u &= g_{\mathrm{D}}, && \text{on } \partial\Omega_{\mathrm{D}}, \\ (\kappa \nabla u - \boldsymbol{c}(u)) \cdot \boldsymbol{n} &= g_{\mathrm{N}}, && \text{on } \partial\Omega_{\mathrm{N}}. \end{aligned} \tag{33}$$

We rewrite it as

$$\begin{aligned} \boldsymbol{q} - \kappa \nabla u &= 0, && \text{in } \Omega, \\ -\nabla \cdot (\boldsymbol{q} - \boldsymbol{c}(u)) &= f, && \text{in } \Omega, \\ u &= g_{\mathrm{D}}, && \text{on } \partial\Omega_{\mathrm{D}}, \\ (\boldsymbol{q} - \boldsymbol{c}(u)) \cdot \boldsymbol{n} &= g_{\mathrm{N}}, && \text{on } \partial\Omega_{\mathrm{N}}. \end{aligned} \tag{34}$$

Here $\boldsymbol{c}(u)$ is a nonlinear flux vector of $u$.

# Formulation

We find $(\boldsymbol{q}_h, u_h, \widehat{u}_h) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (u_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\boldsymbol{q}_h - \boldsymbol{c}(u_h), \nabla w)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{f}}_h \cdot \boldsymbol{n}, w \right\rangle_{\partial \mathcal{T}_h} &= (f, w)_{\mathcal{T}_h}, \\
\left\langle \widehat{\boldsymbol{f}}_h \cdot \boldsymbol{n}, \mu \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega_{\mathrm{D}}} + \langle \widehat{u}_h - g_{\mathrm{D}}, \mu \rangle_{\partial \Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}, \mu \rangle_{\partial \Omega_{\mathrm{N}}},
\end{aligned}
\tag{35}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$, where

$$
\widehat{\boldsymbol{f}}_h = \boldsymbol{q}_h - \boldsymbol{c}(\widehat{u}_h) - \tau(u_h - \widehat{u}_h)\boldsymbol{n} \, . \tag{36}
$$

This completes the definition of the HDG method.

# Stabilization Parameter

The stabilization parameter is chosen as

$$\tau = \tau_{\rm diff} + \tau_{\rm conv} \tag{37}$$

where

$$\tau_{\rm diff} = \frac{\kappa}{\ell}, \qquad \tau_{\rm conv} = |\boldsymbol{c}'(\widehat{u}_h) \cdot \boldsymbol{n}|. \tag{38}$$

Here $\boldsymbol{c}'(\cdot)$ denotes the derivatives of $\boldsymbol{c}(\cdot)$.

In practice, one can choose

$$\tau_{\rm conv} = \text{const} \; > \; |\boldsymbol{c}'(\widehat{u}_h) \cdot \boldsymbol{n}| \; . \tag{39}$$

# Convergence Properties

For **diffusion-dominated** problems, we have

$$\|u - u_h\|_{L^2(\mathcal{T}_h)} \leq C|u|_{H^{k+1}(\mathcal{T}_h)} h^{k+1}, \tag{40}$$

and that

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\mathcal{T}_h)} \leq C|\boldsymbol{q}|_{H^{k+1}(\mathcal{T}_h)} h^{k+1}, \tag{41}$$

and that

$$\|u - u_h^*\|_{L^2(\mathcal{T}_h)} \leq C|u|_{H^{k+2}(\mathcal{T}_h)} h^{k+2}. \tag{42}$$

Both $u_h$ and $\boldsymbol{q}_h$ converge with the **optimal order** $k+1$.
Moreover, $u_h^*$ converges with order $k+2$!

# Weak Formulation

We find $(\boldsymbol{q}_h, u_h, \widehat{u}_h) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (u_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0, \\
-(\nabla\cdot\boldsymbol{q}_h, w)_{\mathcal{T}_h} - (\boldsymbol{c}(u_h), \nabla w)_{\mathcal{T}_h} & \\
+ \langle \boldsymbol{c}(\widehat{u}_h)\cdot\boldsymbol{n} + \tau(u_h - \widehat{u}_h), w\rangle_{\partial\mathcal{T}_h} &= (f, w)_{\mathcal{T}_h}, \\
\langle (\boldsymbol{q}_h - \boldsymbol{c}(\widehat{u}_h))\cdot\boldsymbol{n} - \tau(u_h - \widehat{u}_h), \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_{\mathrm{D}}} & \\
+ \langle (\widehat{u}_h - g_{\mathrm{D}}, \mu\rangle_{\partial\Omega_{\mathrm{D}}} &= \langle g_{\mathrm{N}}, \mu\rangle_{\partial\Omega_{\mathrm{N}}},
\end{aligned} \tag{43}$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$.

We find $(\boldsymbol{q}_h, u_h, \widehat{u}_h) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ such that

$$
\begin{aligned}
(\kappa^{-1}\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (u_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
-(\nabla \cdot \boldsymbol{q}_h, w)_{\mathcal{T}_h} - (\boldsymbol{c}(u_h), \nabla w)_{\mathcal{T}_h} + \langle \tau u_h, w \rangle_{\partial \mathcal{T}_h} & \\
+ \langle \boldsymbol{c}(\widehat{u}_h) \cdot \boldsymbol{n} - \tau \widehat{u}_h, w \rangle_{\partial \mathcal{T}_h} - (f, w)_{\mathcal{T}_h} &= 0, \\
\langle \boldsymbol{q}_h \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega_{\mathrm{D}}} - \langle \tau u_h, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega_{\mathrm{D}}} & \\
\langle -\boldsymbol{c}(\widehat{u}_h) \cdot \boldsymbol{n} + \tau \widehat{u}_h, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega_{\mathrm{D}}} + \langle \widehat{u}_h, \mu \rangle_{\partial \Omega_{\mathrm{D}}} & \\
- \langle g_{\mathrm{D}}, \mu \rangle_{\partial \Omega_{\mathrm{D}}} - \langle g_{\mathrm{N}}, \mu \rangle_{\partial \Omega_{\mathrm{N}}} &= 0,
\end{aligned}
\tag{44}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$.

# Raphson-Newton Method

For any given current solution $(\boldsymbol{q}_h^n, u_h^n, \widehat{u}_h^n)$ we define the residual functionals as

$$r_1^n(\boldsymbol{v}) = (\kappa^{-1}\boldsymbol{q}_h^n, \boldsymbol{v})_{\mathcal{T}_h} + (u_h^n, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{u}_h^n, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, \tag{45}$$

$$\begin{aligned} r_2^n(w) = &-(\nabla\cdot\boldsymbol{q}_h^n, w)_{\mathcal{T}_h} - (\boldsymbol{c}(u_h^n), \nabla w)_{\mathcal{T}_h} + \langle \tau u_h^n, w\rangle_{\partial\mathcal{T}_h} \\ &+ \langle \boldsymbol{c}(\widehat{u}_h^n)\cdot\boldsymbol{n} - \tau\widehat{u}_h^n, w\rangle_{\partial\mathcal{T}_h} - (f, w)_{\mathcal{T}_h}, \end{aligned} \tag{46}$$

$$\begin{aligned} r_3^n(\mu) = &\langle \boldsymbol{q}_h^n\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_{\rm D}} - \langle \tau u_h^n, \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_{\rm D}} \\ &\langle -\boldsymbol{c}(\widehat{u}_h^n)\cdot\boldsymbol{n} + \tau\widehat{u}_h^n, \mu\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_{\rm D}} + \langle \widehat{u}_h^n, \mu\rangle_{\partial\Omega_{\rm D}} \\ &- \langle g_{\rm D}, \mu\rangle_{\partial\Omega_{\rm D}} - \langle g_{\rm N}, \mu\rangle_{\partial\Omega_{\rm N}}. \end{aligned} \tag{47}$$

# Raphson-Newton Method

Update the solution as

$$(\boldsymbol{q}_h^{n+1}, u_h^{n+1}, \widehat{u}_h^{n+1}) := (\boldsymbol{q}_h^{n}, u_h^{n}, \widehat{u}_h^{n}) + (\delta\boldsymbol{q}_h^{n}, \delta u_h^{n}, \delta\widehat{u}_h^{n}) \tag{48}$$

where $(\delta\boldsymbol{q}_h^{n}, \delta u_h^{n}, \delta\widehat{u}_h^{n}) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ is the solution of

$$\begin{aligned}
a^n(\delta\boldsymbol{q}_h^n, \boldsymbol{v}) + b^n(\delta u_h^n, \boldsymbol{v}) + c^n(\delta\widehat{u}_h^n, \boldsymbol{v}) &= -r_1^n(\boldsymbol{v}),\\
d^n(\delta\boldsymbol{q}_h^n, w) + e^n(\delta u_h^n, w) + f^n(\delta\widehat{u}_h^n, w) &= -r_2^n(w),\\
g^n(\delta\boldsymbol{q}_h^n, \mu) + h^n(\delta u_h^n, \mu) + i^n(\delta\widehat{u}_h^n, \mu) &= -r_3^n(\mu),
\end{aligned} \tag{49}$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$.

# Raphson-Newton Method

The bilinear forms are given by

$$
\begin{aligned}
a^n(\boldsymbol{q},\boldsymbol{v}) &= (\kappa^{-1}\boldsymbol{q},\boldsymbol{v})_{\mathcal{T}_h}\\
b^n(u,\boldsymbol{v}) &= (u,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h}\\
c^n(\eta,\boldsymbol{v}) &= -\left\langle\eta,\boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h}\\
d^n(\boldsymbol{q},w) &= -(\nabla\cdot\boldsymbol{q},w)_{\mathcal{T}_h}\\
e^n(u,w) &= -(\boldsymbol{c}'(u_h^n)u,\nabla w)_{\mathcal{T}_h} + \left\langle\tau u,w\right\rangle_{\partial\mathcal{T}_h},\\
f^n(\eta,w) &= \left\langle(\boldsymbol{c}'(\widehat{u}_h^n)\cdot\boldsymbol{n}-\tau)\eta,w\right\rangle_{\partial\mathcal{T}_h},\\
g^n(\boldsymbol{q},\mu) &= \left\langle\boldsymbol{q}\cdot\boldsymbol{n},\mu\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_{\mathrm{D}}}\\
h^n(u,\mu) &= -\left\langle\tau u,\mu\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_{\mathrm{D}}},\\
i^n(\eta,\mu) &= \left\langle(\tau-\boldsymbol{c}'(\widehat{u}_h^n)\cdot\boldsymbol{n})\eta,\mu\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_{\mathrm{D}}} + \left\langle\eta,\mu\right\rangle_{\partial\Omega_{\mathrm{D}}}.
\end{aligned}
\tag{50}
$$

Here the prime denotes the partial derivatives.

# Matrix System

At each Newton iteration, we solve the following system

$$\begin{bmatrix} \mathbb{A}^n & \mathbb{B}^n & \mathbb{C}^n \\ \mathbb{D}^n & \mathbb{E}^n & \mathbb{F}^n \\ \mathbb{G}^n & \mathbb{H}^n & \mathbb{I}^n \end{bmatrix} \begin{bmatrix} \delta\mathbb{Q}^n \\ \delta\mathbb{U}^n \\ \delta\widehat{\mathbb{U}}^n \end{bmatrix} = - \begin{bmatrix} \mathbb{R}_1^n \\ \mathbb{R}_2^n \\ \mathbb{R}_3^n \end{bmatrix} \tag{51}$$

where $\delta\mathbb{Q}^n$, $\delta\mathbb{U}^n$ and $\delta\widehat{\mathbb{U}}^n$ are the vectors of degrees of freedom of $\delta\boldsymbol{q}_h^n$, $\delta u_h^n$ and $\delta\widehat{u}_h^n$, respectively.

Again we note that the matrix

$$\begin{bmatrix} \mathbb{A}^n & \mathbb{B}^n \\ \mathbb{D}^n & \mathbb{E}^n \end{bmatrix} \tag{52}$$

is **block-diagonal**. Hence, we can eliminate $(\delta\mathbb{Q}^n, \delta\mathbb{U}^n)$ to obtain a reduced system for $\widehat{\mathbb{U}}^n$.

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